

Equilibration of two non-extensive subsystems in a parton cascade model

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Abstract

We study the process of equilibration between two non-extensive subsystems in the framework of a particular non-extensive Boltzmann equation. We have found that even subsystems with different non-extensive properties achieve a common equilibrium distribution.

Key words: non-extensive, statistics, Boltzmann equation

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1. Introduction

Power-law tailed spectra occur in a wide range of physical phenomena, re-association in folded proteins [1], fluxes of cosmic rays [2], turbulence [3], finance and economics [4], electron-positron annihilation [5], motion of Hydra cells [6], epilepsy [7], linguistics [8], nuclear physics [9], astrophysics [10], field theories and cosmology [11], scientific citations [12], distributions of individual succes of musicians [13], urban agglomerations [14], internet phenomena [15], phase transformations [16], in algorithms for global optimization (e.g., the generalized simulated annealing) and related computational methods [17] and information theory [18].

We are particularly interested in heavy ion collisions where cut power-law distributions describe transverse momentum (p_T) spectra at low and intermediate values [19]. The conventional approach [20] to the distribution function fits a Boltzmann-Gibbs exponential characterized by a single temperature and the deviation from the exponential at the tail

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is attained to non-equilibrium effects. A transverse expansion distorts the locally thermal spectra; at low p_T by suppression, at high p_T by a blue shift factor in the temperature.

Some further attempts have been made to interpret power-law tailed spectra as equilibrium phenomena for the whole p_T range: Non-extensive thermodynamics predict such distributions [9,21]. It has been shown, that dynamicses among (quasi-) particles can be constructed, leading to power-law tailed stationary distributions [22]. This parton cascade model, referred to as non-extensive Boltzmann equation (NEBE), generates stationary states subject to non-extensive kinetic energy addition rules. We have shown that an infinite class of equilibrium distributions can be established depending on the way we generalize the Boltzmann equation [23]. In this paper we point out that cut power-law distributions play a special role among all possible functions: The rule applied in the non-extensive Boltzmann equation leading to such distributions is the leading term in a low-energy expansion of a general associative rule.

A crucial question towards non-extensive thermodynamics has been discussed in [24], namely how do two non-extensive thermodynamical systems thermalize, if at all. In this article we aim to answer this question by performing momentum-space simulations within the NEBE model.

2. Non-Extensive Boltzmann Equation (NEBE)

In the non-extensive extension of the Boltzmann equation we keep the original factorizing form for the two-body distributions,

$$\frac{\partial}{\partial t} f_1 = \int_{234} w_{1234} (f_3 f_4 - f_1 f_2), \quad (1)$$

but in the transition rate

$$w_{1234} = M_{1234}^2 \cdot \delta((\mathbf{p}_1 + \mathbf{p}_2) - (\mathbf{p}_3 + \mathbf{p}_4)) \cdot \delta(h(E_1, E_2) - h(E_3, E_4)), \quad (2)$$

the two-body energy composition is generalized to a rule $h(E_1, E_2)$, which is not necessarily the simple addition. There can be many physical sources for this deformation of the kinetic energy addition rule; the most general being a pair interaction due to a potential whose value differs before and after the two body collision. This way corrections occur, which (e. g. by using a virial theorem) may be expressible in terms of the individual kinetic energies of the colliding subsystems (particles). Here $E_i = \sqrt{\mathbf{p}_i^2 + m_i^2}$ are the free kinetic energies of relativistic particles with mass m_i . In a heavy-ion collision they are regarded as the asymptotic energies detected after desintegration of the system.

The energy composition rule $h(E_1, E_2)$ contains contributions stemming from the pair interaction. It is not trivial whether these can always be divided to one-particle contributions, supporting a quasi-particle picture. In the followings we demonstrate that under quite general assumptions about the function $h(x, y)$ the division of the total energy among free particles can be done.

We assume that the generalized energy sum is associative,

$$h(h(x, y), z) = h(x, h(y, z)). \quad (3)$$

Then due to a mathematical theorem [25] a strict monotonic function $X(h)$ maps the energy composition rule to additivity

$$X(h) = X(x) + X(y). \quad (4)$$

This solution (4) of the functional equation (3) is unique up to a constant multiplicative factor. In this case the stationary solution of the non-extensive Boltzmann equation is given by

$$f(\mathbf{p}) = \frac{1}{Z} e^{-X(E)/T}, \quad (5)$$

and $X(E)$ is regarded as the energy of a quasi-particle.

One finds several examples for the energy composition rule $h(x, y)$. Some of them lead to thermodynamicses where the entropy formula or the distribution function is familiar: e.g. $h(x, y) = axy$ leads to the Rényi-entropy $S_R = \frac{1}{q-1} \ln \int f^q$, with $s = \frac{1}{1-q} \frac{f^q}{f}$, $q = 1 - aT$, and $X(E) = \frac{1}{a} \ln(aE)$. The distribution function is given by $f(E) = \frac{1}{Z} (aE)^{-1/(aT)}$.

The energy composition rule

$$h(x, y) = x + y + axy \quad (6)$$

has the solution

$$X(E) = \frac{1}{a} \ln(1 + aE), \quad (7)$$

and leads to the cut power-law stationary distribution

$$f(E) = \frac{1}{Z} (1 + aE)^{-1/aT}. \quad (8)$$

(cf. for $q = 1 - aT$ the q -Tsallis distribution emerges [23])

This form is the next to leading order expansion of a general associative rule for low energy: By Taylor expanding eq. (4) for low x, y and h values and requiring $h(x, 0) = x$, $h(0, y) = y$ one arrives at

$$h(x, y) = x + y - \frac{X''(0)}{X'(0)} xy + \dots \quad (9)$$

We conclude that the Tsallis-type composition rule with $a = -X''(0)/X'(0)$ is generic for leading order non-extensive effects at low energy.

3. Non-extensive parton cascade

In order to investigate the equilibration of non-extensive systems we start with two subsystems, equilibrated separately. In order to prepare these systems the non-extensive Boltzmann equation is solved numerically in a parton cascade simulation [22]. We use several different initial momentum distributions, and then make random binary collisions between randomly chosen pairs of particles. By doing so we apply the rules

$$X(E_1) + X(E_2) = X(E_3) + X(E_4), \quad (10)$$

$$\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_3 + \mathbf{p}_4. \quad (11)$$

In each step of the simulation we select two particles to collide. Then we find the value for the new momentum of the first particle (\mathbf{p}_3) satisfying the above constraints but otherwise random. Then applying eq. (11) we calculate the momentum of the second outgoing particle (\mathbf{p}_4). In these particular simulations, this paper reports about, we use the free dispersion relation for massless particles ($E_i(\mathbf{p}_i) = |\mathbf{p}_i|$). We proceed with the next

collision repeating the above steps with a new randomly chosen pair of particles. A typical simulation includes $10^6 - 10^7$ collisions among $10^5 - 10^6$ particles. After $3 - 5$ collisions per particle on the average, the one-particle distribution approaches its stationary form sufficiently.

The following quantities are conserved during the simulation:

$$X(E_{tot}) = \sum_{i=1}^N X(E_i), \quad \mathbf{P} = \sum_{i=1}^N \mathbf{p}_i, \quad N = \sum_{i=1}^N 1. \quad (12)$$

We use eq. (6) for the energy composition rule, where a is called the non-extensivity parameter. Our model reconstructs the traditional Boltzmann-Gibbs thermodynamics in the limit of $a = 0$. More details can be found in [22].

We perform simulations on different systems with particle numbers N_1 and N_2 , total (quasi-)energies $X(E_1)$ and $X(E_2)$ and non-extensivity parameters a_1 and a_2 . We evolve these systems until they reach their stationary states. The unified system is taken as an initial state with $N = N_1 + N_2$ particles.

Now we have three non-extensivity parameters a_1 , a_2 and a_{12} , describing the three possible interaction types in the composed system. a_1 is the parameter of the collisions among particles in the first subsystem, similarly a_2 corresponds to the second subsystem, and finally a_{12} is the non-extensivity parameter in a collision between a particle from the first subsystem and a particle from the second subsystem. We investigate cases with common and different a parameters.

4. Results

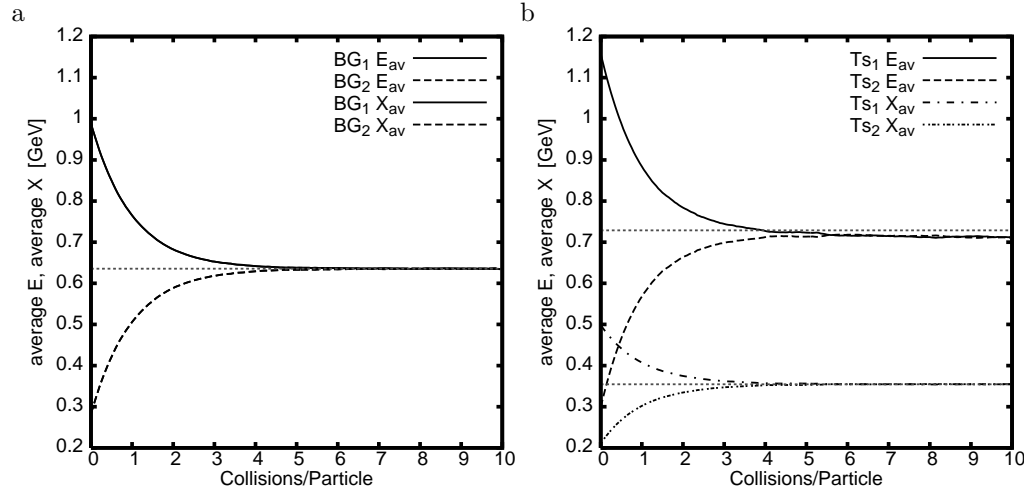


Fig. 1. *a*, Equilibration of two Boltzmann-Gibbs systems ($a_1 = a_2 = a_{12} = 0$) ; *b*, Equilibration of two Tsallis-type non-extensive systems ($a_1 = a_2 = a_{12} = 2$)

Our results show that the subsystems equilibrate, in the final state of the composed system they have a common stationary distribution.

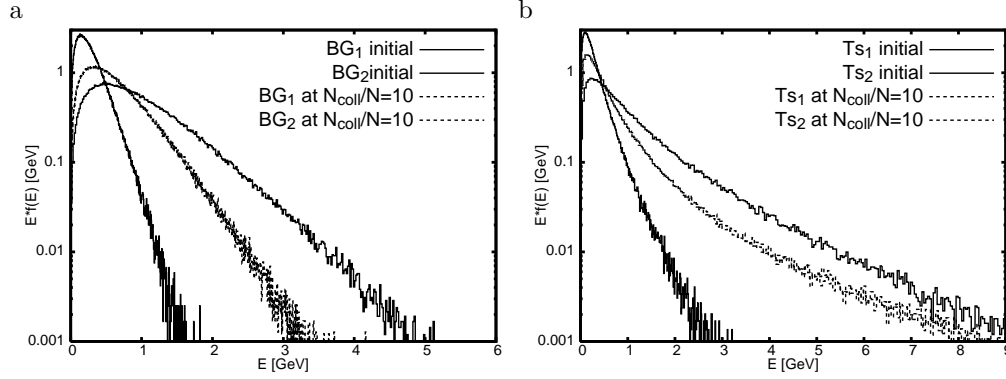


Fig. 2. *a*, Equilibration of two Boltzmann-Gibbs systems ($a_1 = a_2 = a_{12} = 0$) ; *b*, Equilibration of two Tsallis-type non-extensive systems ($a_1 = a_2 = a_{12} = 2$)

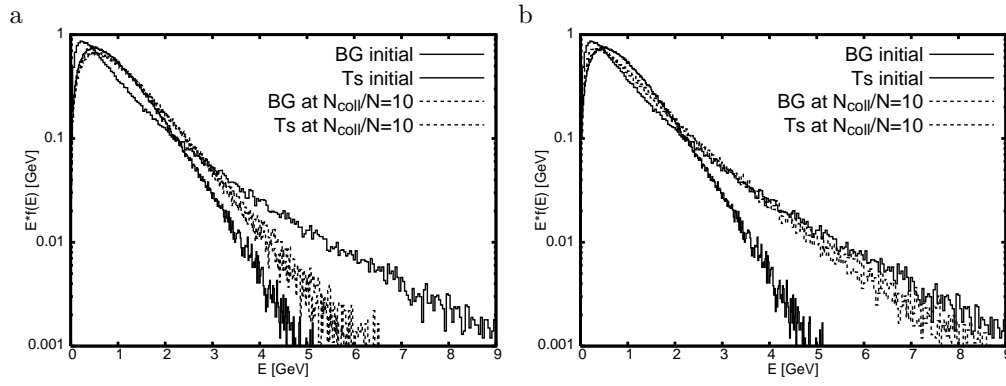


Fig. 3. *a*, Equilibration of a Boltzmann-Gibbs and a Tsallis-type non-extensive system ($a_1 = 0, a_2 = 2$ and $a_{12} = 0$) ; *b*, Equilibration of a Boltzmann-Gibbs and a Tsallis-type non-extensive system ($a_1 = 0, a_2 = 2$ and $a_{12} = 2$)

We present examples with different initial conditions. In all of these simulations the particle numbers are the same for each subsystem, $N_1 = N_2 = 250\,000$. The number of collisions in one simulation is $N_{coll} = 5\,000\,000$, so $N_{coll}/(N_1 + N_2) = 10$ collisions happen per particle. Figure 1 shows the average energy and quasi-energy versus the number of collisions per particle. On each part of the figures 2 and 3 four curves show the initial (continuous line) and final (dashed line) energy distributions of subsystems 1 and 2. It is hard to distinguish the final states of the subsystems, because the corresponding distributions are very near to each other. Within numerical uncertainties they have a common stationary energy distribution.

Figures 1*a* and 2*a* show the equilibration of two Boltzmann-Gibbs systems, where a_1, a_2 and a_{12} are equal to 0. This simulation was done for test purposes. As expected, in the final state the two subsystems have a common stationary energy distribution with a common temperature. The energy content per particle of the composed system became the arithmetic mean of the respective initial values. On figure 1*a* the curves for average $X_{av} = \sum_i X(E_i)/N$ coincides with the curve for $E_{av} = \sum_i E_i/N$, which comes from the

definition of the quasi-particle energy (cf. eq. (7) with $a = 0$).

The equilibration of two Tsallis-type non-extensive systems with $a_1 = a_2 = a_{12} = 2$ is shown on figures 1b and 2b. The initial quasi-energy contents are different but at the end this difference disappears. X_{av} becomes the arithmetic mean in the composed system. This is illustrated by the constant for $(X_{1,av}^{initial} + X_{2,av}^{initial})/2$. The final average kinetic energy per particle, $E_{1,av}^{final} = E_{2,av}^{final} \neq (E_{1,av}^{initial} + E_{2,av}^{initial})/2$, is slightly below the average of the initial energies.

On the figures 3a and 3b we show the equilibration between two systems with different non-extensivity parameters: A Boltzmann-Gibbs system ($a = 0$) and another one with $a = 2$. In 3a the non-extensivity parameter for a collision between particles coming from different systems is taken to be $a_{12} = 0$. In 3b this parameter is given by $a_{12} = 2$. In these cases neither X_{av} nor E_{av} approaches to the arithmetic mean of the initial X_{av} or E_{av} values (actually X_{av} cannot be defined for the composed system). The final common stationary energy distributions are between the initial energy distributions of the original subsystems (with $a_1 = 0$ and $a_2 = 2$); in case a ($a_{12} = 0$) it is closer to the Boltzmann-Gibbs system, in case b ($a_{12} = 2$) it is closer to the original non-extensive system.

5. Conclusions

We studied the equilibration between two non-extensive systems in the framework of a relativistic parton cascade model, NEBE. We have demonstrated earlier that these systems have non-exponential equilibrium distributions in terms of the free particle kinetic energy, which is dependent on the chosen type of non-extensivity (or equivalently on the details of energy sharing in the two-particle collisions). The composition of different non-extensive systems resulted in a common equilibrium distribution.

We have pointed out that the non-extensive energy addition rule eq. (6), leading to a cut power-law distribution in equilibrium, is the next to leading order low-energy expansion of a general associative rule.

We coupled two subsystems after a separate equilibration both by additive and super-additive rules. The subsystems tend to achieve a common distribution. This process coincides with the equilibration of the temperatures in the case of equal non-extensivity parameters. In the case of different non-extensivity parameters, however, neither the average of original one-particle energies nor the average of the quasi-particle energies is observed. In this case the associativity of the energy composition rule is violated, so actually a quasi-particle energy can not be defined.

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